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ASYMPTOTIC ANALYSIS OF SUMS OF POWERS OF MULTINOMIAL COEFFICIENTS: A SADDLE POINT APPROACH

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Abstract
We utilize the saddle point method for obtaining the asymptotic growth of the sums of powers of multinomial coefficients. We give an overview of the many places that such sums of powers of multinomial coefficients appear in the mathematical literature. Our proof methodology follows an analytic (complex-valued) approach, including usage of the saddle point method.

Dedicated to Dr. Daniel D. Bonar, the George R. Stibitz Distinguished Professor Emeritus in Mathematics and Computer Science at Denison University, a professor, mentor, and dear friend, in celebration of his 50 years of teaching Mathematics.

1. Introduction and Motivation

We define

$$a_{m,k}(n) := \sum_{i_1+i_2+\cdots+i_m=n} \binom{n}{i_1, i_2, \ldots, i_m}^k,$$

and

$$b_{m,k}(n) := \frac{a_{m,k}(n)}{(n!)^k}.$$

We believe that the asymptotic properties of these sequences have only been precisely analyzed in special cases. Due to the fundamental nature of these integer

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sequences, we desired to make a comprehensive characterization of the asymptotic growth of these integer sequences, as $n \to \infty$, for any (fixed) positive integers m and k. Our main result is the following:

Theorem 1. *For (fixed) positive integers m and k, as $n \to \infty$, the first-order asymptotic growth of $b_{m,k}(n)$ is*

$$b_{m,k}(n) \sim \left(\frac{em}{n}\right)^{kn} \sqrt{\frac{1}{2\pi n k^{m-1}} \left(\frac{m}{2\pi n}\right)^{(k-1)m}}. \tag{1}$$

Stirling's approximation is $n! \sim \sqrt{2\pi n}\,(n/e)^n$. Taking a power of k in Stirling's approximation, and then multiplying the result on both sides of Equation (1), our theorem immediately yields the following corollary.

Corollary 1. *For (fixed) positive integers m and k, as $n \to \infty$, the first-order asymptotic growth of $a_{m,k}(n)$ is*

$$a_{m,k}(n) \sim m^{kn} \sqrt{\frac{m^{k-1}}{k^{m-1}} \left(\frac{m}{2\pi n}\right)^{(k-1)(m-1)}}. \tag{2}$$

These integer sequences are intimately connected with hypergeometric functions. The general family $a_{m,k}(n)$ appears, for instance, in Barrucand [2]. The asymptotic growth of $a_{2,k}(n)$ has been known for almost a century [15], and perhaps longer.

In the case $k = 1$, the values of $a_{m,k}(n)$ are simply the powers of m, namely, $a_{m,1}(n) = m^n$.

The $k = 2$ case has myriad interpretations. Consider a uniform planar random walk, starting at the origin and consisting of m steps, each of length 1, and each taken in a random direction (independent of all previous choices of direction). Then $a_{m,2}(n)$ is the $2n$th moment of the distance from the origin after such a walk. (See Richards and Cambanis [16], as well as Borwein et al. [4].)

Richmond and Shallit [18] also point out that $a_{m,k}(n)$ enumerates the abelian kth powers. An *abelian kth power* consists of a sequence of k blocks, each of length n. The first block consists of n letters selected from an alphabet with m letters (with repetitions allowed). Each of the remaining $k - 1$ blocks is a permutation of the letters in the first block.

Bernstein and Lange [3] use the family of sequences $a_{m,2}(n)$; they make a "connection ... between anticollision factors and sums of squares of multinomials." Verrill [20] gives an explicit recurrence for $a_{m,2}(n)$ for each fixed m.

The integers $a_{4,2}(n)$ are known as the Domb numbers; they enumerate the $2n$-step polygons on a diamond lattice (see [8] and also OEIS #A002895). The sequence $a_{2,2}(n) = \binom{2n}{n}$ consists of the central binomial coefficients, which are utilized in many applications; see OEIS #A000984.

The sequence $a_{2,3}(n)$ consists of the Franel numbers, which have dozens of applications. See OEIS #A000172 as a starting point for the vast literature on this sequence.

The sequences $a_{m,k}(n)$ were recently used in their full generality in a manuscript by Tao [19]; see especially Tao's Theorem 7 (and its proof), in which he analyzes "the mean number of occurrences of [a pattern] p in the abelian sense in a word of length n over an alphabet of $m \geq 4$ letters". See also Tao's Corollaries 2 and 3.

Remark 1. We summarize special cases that were already proved in the literature.

$k = 2$: The asymptotic growth of $a_{m,2}(n)$, for fixed m, as $n \to \infty$, was established by Richmond and Rousseau [17] using an approach with complex functions, similar to the methodology that we utilize here. (An alternative approach to establishing the asymptotic growth was later discovered by Richmond and Shallit [18].) Richmond and Rousseau [17] analyzed one Hayman-admissible function. (For comparison, and to note why the general analysis is more complicated, we emphasize that, for general k, we compare the behavior of $2P+1$ functions, where $P = 2\lfloor (k+2)/8 \rfloor + 1$; see our Equation (5).)

$m = 2$: The asymptotic growth of $a_{2,k}(n)$ was analyzed as early as 1925; see Polya and Szegő [15, Problem 40, p. 55, of the 1972 English edition]. See also Farmer and Leth [9] and Wilson [21] for more recent discussions.

Our result in Corollary 1 is a full generalization of these cases. We analyze the asymptotic growth of $a_{m,k}(n)$ for *any* (fixed) positive integers m and k, as $n \to \infty$.

2. Notation and Background

Since $\binom{n}{i_1, i_2, \ldots, i_m} = \frac{n!}{i_1! i_2! \cdots i_m!}$, it follows that

$$b_{m,k}(n) = \sum_{i_1 + i_2 + \cdots + i_m = n} \frac{1}{(i_1!)^k} \frac{1}{(i_2!)^k} \cdots \frac{1}{(i_m!)^k}.$$

Now we define

$$f_k(z) := \sum_{i=0}^{\infty} \frac{z^i}{(i!)^k},$$

and we observe that

$$\sum_{n=0}^{\infty} b_{m,k}(n) z^n = (f_k(z))^m.$$

The function $f_k(z)$ will be a central object of study, in our proof methodology.

We use the Pochhammer symbol, $(\alpha)_n := (\alpha)(\alpha+1)\cdots(\alpha+n-1)$ and also the notation for generalized hypergeometric series:

$${}_pF_q\left({a_1, \ldots, a_p \atop b_1, \ldots, b_q}; z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{z^j}{j!}.$$

We can frame the analysis of $f_k(z)$ in terms of hypergeometric series. When k is a positive integer, we have

$$f_k(z) = {}_0F_{k-1}\left({\emptyset \atop 1,\ldots,1}; z\right). \tag{3}$$

For this interpretation of $f_k(z)$, we have $p = 0$, and therefore there are no a's. Regarding the b's, the sequence of 1's has length $k-1$, so $q = k-1$. To analyze the asymptotic behavior of $f_k(z)$, we use Askey and Daalhuis [1, Section 16.11]. Following their notation, in (16.11.3), their κ is equivalent to our k, and their ν is equal to our $-(k-1)/2$. Unfortunately, however, we cannot use their formulation from (16.11.4) and (16.11.5), because in our situation, the b_j's are repeated; this causes the denominator of (16.11.5) to be zero. An alternative formulation must be used [6]. Curious readers can also compare with the notation from Paris and Kaminski [12, section 2.3], where many of the relevant details are explained. As further background reading for the interested reader, one might consider the earlier treatments found in [5, Section 12] and [13, Section 2.3].

3. Proofs

Our overall goal is to treat the functions under study as complex-valued objects, and then to use the saddle point method to retrieve asymptotic information about $b_{m,k}(n)$. For a very readable discussion about such methods, we suggest Flajolet and Sedgewick [10, Chapter VIII]. We have

$$b_{m,k}(n) = \frac{1}{2\pi\mathbf{i}} \int_{\Omega} \frac{(f_k(z))^m}{z^{n+1}}\,dz, \tag{4}$$

where Ω is a closed contour in the counterclockwise direction about the origin. To use the saddle point method, we focus on a contour that is a circle centered at the origin. So we use $\Omega := \{z = \rho e^{\mathbf{i}\theta} \mid -\pi \le \theta \le \pi\}$; in particular, we use a contour Ω that only depends on the choice of ρ.

Before we use the saddle point method, it is helpful to decompose $f_k(z) = \sum_{i=0}^{\infty} z^i/(i!)^k$. We need to be mindful of the asymptotic behavior of $1/(n!)^k$ for large n. For this purpose, we replace the discrete $n!$ by the continuous $\Gamma(s+1)$, and we expand as a series. It is well known that there are unique constants $c_k^{(j)}$ such that

$$\frac{1}{(\Gamma(s+1))^k} \sim \frac{k^{ks+k/2}}{(2\pi)^{(k-1)/2}} \sum_{j=0}^{\infty} \frac{c_k^{(j)}}{\Gamma(ks + \frac{k+1}{2} + j)}, \qquad \text{as } s \to \infty.$$

As for the values of these constants, in the words of Paris and Kaminski, "their actual evaluation turns out to be the most difficult part of the theory" [12, p. 57].

The first several values of $c_k^{(j)}$ are:

$$c_k^{(0)} = 1, \qquad c_k^{(1)} = \frac{k^2-1}{24}, \qquad c_k^{(2)} = \frac{(k^2-1)(k^2+23)}{(24^2)(2)},$$

etc. Paris and Kaminski [12, p. 58] prove the approximation

$$f_k(z) \sim \sum_{r=-P}^{P} E_{k,r}(z), \tag{5}$$

as $|z| \to \infty$, where the functions $E_{k,r}(z)$ are defined as

$$E_{k,r}(z) := \frac{\exp(k\, z^{1/k} e^{2\pi \mathbf{i} r/k})}{\sqrt{k}\,(2\pi)^{(k-1)/2} z^{(k-1)/(2k)} e^{2\pi \mathbf{i} r(k-1)/(2k)}} \sum_{j=0}^{\infty} \frac{c_k^{(j)}}{k^j z^{j/k} e^{2\pi \mathbf{i} rj/k}}. \tag{6}$$

As in Paris and Kaminski's exposition, "P is chosen such that $2P+1$ is the smallest odd integer satisfying $2P+1 > \frac{1}{2}\kappa$." In our case, $\kappa = k$. An elementary calculation shows that, for our analysis, the relevant value of P is exactly $P = 2\lfloor (k+2)/8 \rfloor + 1$.

Now we need to precisely analyze the behavior of each function $E_{k,r}(z)$.

Lemma 1. *Consider $r \neq 0$ with $-P \le r \le P$. Let $z = \rho e^{\mathbf{i}\theta}$ where $-\pi \le \theta \le \pi$. Then we have*

$$|E_{k,r}(\rho e^{\mathbf{i}\theta})| = O\left(\frac{\exp\left(k\,\rho^{1/k}\cos(\pi/k)\right)}{\rho^{(k-1)/(2k)}}\right)$$

as $\rho \to \infty$.

Proof. We consider the exponential term $\exp(k\, z^{1/k} e^{2\pi \mathbf{i} r/k})$ in the numerator of $E_{k,r}(z)$. Since $z = \rho e^{\mathbf{i}\theta}$, it follows that

$$\exp\left(k\, z^{1/k} e^{2\pi \mathbf{i} r/k}\right) = \exp\left(k \rho^{1/k} e^{\mathbf{i}(\theta + 2\pi r)/k}\right).$$

Now taking the modulus, we have

$$\left|\exp\left(k\, z^{1/k} e^{2\pi \mathbf{i} r/k}\right)\right| = \left|\exp\left(k\rho^{1/k} \cos\left((\theta + 2\pi r)/k\right)\right)\right|.$$

Since $r \neq 0$ and $-P \le r \le P$, then we have $0 < |r| \le k$. Only using the fact that $0 < |r| \le k$, it follows immediately from basic trigonometry that $\cos((\theta + 2\pi r)/k) \le \cos(\pi/k)$ for all $-\pi \le \theta \le \pi$. So it follows that

$$\left|\exp\left(k\, z^{1/k} e^{2\pi \mathbf{i} r/k}\right)\right| \le \exp\left(k\,\rho^{1/k}\cos(\pi/k)\right).$$

Now the lemma follows immediately, by inspecting the moduli of $E_{k,r}(\rho e^{\mathbf{i}\theta})$. □

When $z = \rho e^{\mathbf{i}\theta}$ with $\theta = \pm\pi$, we observe that

$$\exp\left(k\,\rho^{1/k}\cos(\pi/k)\right) \le \left|\exp(k\, z^{1/k})\right|.$$

It follows by Lemma 1 that, for $r \neq 0$ with $-P \leq r \leq P$, and $-\pi \leq \theta \leq \pi$, we have

$$\left|E_{k,r}(\rho e^{\mathbf{i}\theta})\right| = O\big(\left|E_{k,0}(\rho e^{\pm\mathbf{i}\pi})\right|\big).$$

In other words, $|E_{k,r}(\rho e^{\mathbf{i}\theta})|$ (for $r \neq 0$ and for any z on Ω) is dominated by the value of $|E_{k,0}(z)|$ with z at either endpoint of Ω.

Moreover, the value of $|E_{k,0}(\rho e^{\mathbf{i}\theta})|$ increases monotonically (for fixed ρ) as θ decreases from π down to 0, and similarly, increases monotonically as θ increases from $-\pi$ up to 0. (This can be checked by some tedious algebra; see, e.g., [10, Ch. VIII].) When considering

$$\frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{(E_{k,0}(z))^m}{z^{n+1}}\,dz,$$

we will see that it suffices to consider only the portion of Ω corresponding to $-\theta_0 \leq \theta \leq \theta_0$ for (say) $\theta_0 = n^{-4/9}$. (We will clarify the reasoning for 4/9 in Section 3.1.2.) The remainder of the contribution from the rest of the contour Ω will have asymptotically lower order. Moreover, $|E_{k,r}(\rho e^{\mathbf{i}\theta})|$ is bounded above by $|E_{k,0}(\rho e^{\pm\mathbf{i}\pi})|$, i.e., by the size of $|E_{k,0}(z)|$ at the endpoints of Ω. Therefore, the contribution from the $E_{k,r}(z)$ for $r \neq 0$ can safely be ignored, when computing the first-order asymptotic growth of $b_{m,k}(n)$.

For this reason, we only use $E_{k,0}$ (and not the other $E_{k,r}$'s) when calculating the asymptotic behavior of $b_{m,k}(n)$. In other words, Equation (4) can be simplified to

$$b_{m,k}(n) \sim \frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{(E_{k,0}(z))^m}{z^{n+1}}\,dz.$$

3.1. Saddle Point Method

Now we use the saddle point technique. See de Bruijn [7], Good [11], or Flajolet and Sedgewick [10] for a description of this technique. The basic motivation of the saddle point method is to deform the contour of integration Ω to a radius ρ, in such a way that the contribution to the integral representation of $b_{m,k}(n)$ is well approximated by a Gaussian integral over the range $-\theta_0 < \theta < \theta_0$, i.e., near the large, positive portion of the real axis. The integral representation of $b_{m,k}(n)$ over the remainder of Ω can then be truncated (called "pruning the tails") because it is much smaller, as compared to the aforementioned Gaussian integral.

If we define

$$L_{k,j}(\rho) := \partial_u^j \ln(E_{k,0}(\rho e^u))\big|_{u=0}\,,$$

then, by Cauchy's theorem, we have

$$\begin{aligned} b_{m,k}(n) &\sim \frac{1}{2\pi\mathbf{i}} \int_{\Omega} \frac{(E_{k,0}(z))^m}{z^{n+1}}\, dz \\ &= \frac{1}{\rho^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big(m\big(\ln(E_{k,0}(\rho)) + \mathbf{i}L_{k,1}(\rho)\theta \\ &\qquad - L_{k,2}(\rho)\theta^2/2 - \mathbf{i}L_{k,3}(\rho)\theta^3/6 + \dots \big) - n\mathbf{i}\theta \Big)\, d\theta. \end{aligned} \tag{7}$$

3.1.1. Finding the Location of the Saddle Point

To find the location ρ of the saddle point, we need:

$$\frac{\partial}{\partial \rho}\Big(m \ln(E_{k,0}(\rho)) - n \ln(\rho) \Big) = 0,$$

and, therefore,

$$m \frac{E'_{k,0}(\rho)}{E_{k,0}(\rho)} - \frac{n}{\rho} = 0.$$

Thus, the saddle point ρ is the unique root of smallest modulus of

$$m\rho E'_{k,0}(\rho) - nE_{k,0}(\rho) = 0. \tag{8}$$

Since $L_{k,1}(\rho) := \partial_u \ln(E_{k,0}(\rho e^u))|_{u=0}$, then it follows that:

$$L_{k,1}(\rho) = \frac{\rho\, E'_{k,0}(\rho)}{E_{k,0}(\rho)}. \tag{9}$$

Combining (8) and (9), we see that the saddle point ρ satisfies

$$m\, E_{k,0}(\rho) L_{k,1}(\rho) - nE_{k,0}(\rho) = 0,$$

but $E_{k,0}(\rho) \neq 0$. So dividing by $E_{k,0}(\rho)$ and multiplying by $\mathbf{i}\theta$, we obtain

$$m\mathbf{i}L_{k,1}(\rho)\theta - n\mathbf{i}\theta = 0.$$

Therefore, Equation (7) simplifies, when ρ is chosen at the saddle point, to become

$$b_{m,k}(n) \sim \frac{1}{\rho^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big(m\big(\ln(E_{k,0}(\rho)) - L_{k,2}(\rho)\theta^2/2 - \mathbf{i}L_{k,3}(\rho)\theta^3/6 + \dots \big) \Big)\, d\theta,$$

or, more simply,

$$b_{m,k}(n) \sim \frac{1}{\rho^n} \frac{(E_{k,0}(\rho))^m}{2\pi} \int_{-\pi}^{\pi} \exp\Big(m\big(- L_{k,2}(\rho)\theta^2/2 - \mathbf{i}L_{k,3}(\rho)\theta^3/6 + \dots \big) \Big)\, d\theta.$$

We have

$$E_{k,0}(z) = \frac{\exp(k\, z^{1/k})}{\sqrt{k}\, (2\pi)^{(k-1)/2} z^{(k-1)/(2k)}} \big(1 + \Theta(z^{-1/k}) \big), \tag{10}$$

as $z \to \infty$ along the positive real axis. Solving for $E'_{k,0}(z)$, and then using Equation (8), and only preserving the terms up to first order (as $z \to \infty$ along the positive real axis), we obtain $m\rho(z^{(1-k)/k}) - n = 0$, and we conclude that the location of the saddle point is

$$\rho \sim (n/m)^k, \quad n \to \infty. \tag{11}$$

3.1.2. Splitting the Contour

We recall $L_{k,j}(\rho) := \partial_u^j \ln(E_{k,0}(\rho e^u))|_{u=0}$. Using (10), we obtain, for all $j \geq 1$,

$$L_{k,j}(\rho) = \partial_u^j \ln(E_{k,0}(\rho e^u))|_{u=0} \sim \frac{\rho^{1/k}}{k^{j-1}}. \tag{12}$$

Next, we choose a splitting value θ_0 with the property that $mL_{k,2}(\rho)\theta_0^2 \to \infty$ and $mL_{k,3}(\rho)\theta_0^3 \to 0$ as $n \to \infty$ (i.e., as $\rho \to \infty$). These two conditions translate to:

$$m\Big(\frac{\rho^{1/k}}{k}\Big)\theta_0^2 \to \infty, \qquad \text{and} \qquad m\Big(\frac{\rho^{1/k}}{k^2}\Big)\theta_0^3 \to 0.$$

The location of the saddle point is $\rho \sim (n/m)^k$, so the previous equations become

$$m\Big(\frac{n/m}{k}\Big)\theta_0^2 \to \infty, \qquad \text{and} \qquad m\Big(\frac{n/m}{k^2}\Big)\theta_0^3 \to 0,$$

so it suffices to have $n\theta_0^2 \to \infty$ and $n\theta_0^3 \to 0$. So we need an angle $\theta_0 = n^\alpha$ for $-1/2 < \alpha < -1/3$. For instance, we can use $\theta_0 = n^{-4/9}$.

3.1.3. Pruning the Tails

To prune the tails, let $\mathcal{T} := (-\pi, -\theta_0) \cup (\theta_0, \pi)$ denote the tail region of the contour. We compute

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{\mathcal{T}} \frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = \frac{1}{\rho^n}\frac{1}{2\pi}O\bigg(\Big(\frac{\exp\big(k\,\rho^{1/k}\cos(\theta_0/k)\big)}{\rho^{(k-1)/(2k)}}\Big)^m\bigg), \tag{13}$$

but the saddle point is located at distance $\rho \sim (n/m)^k$ and we are using $\theta_0 = n^{-4/9}$, so $\cos(\theta_0/k) = O(1 - (\theta_0/k)^2/2)$. Therefore we can rewrite Equation (13) as

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{\mathcal{T}} \frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = \frac{1}{(n/m)^{kn}}\frac{1}{2\pi}O\bigg(\frac{\exp\big(kn\big(1-(n^{-4/9}/k)^2/2\big)\big)}{(n/m)^{m(k-1)/2}}\bigg)$$

or even more simply as

$$\left|\frac{1}{\rho^n}\int_{\mathcal{T}} \frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = O\big((em/n)^{kn}\exp(-n^{1/9}/(2k))(m/n)^{(k-1)m/2}\big).$$

Looking ahead, for comparison to the final asymptotic behavior of $b_{m,k}(n)$ in Equation (14), and noting that k and m are held constant, we see that $\exp(-n^{1/9}/(2k))$ decreases much faster than $\sqrt{1/n}$, and thus

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{\mathcal{T}}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = o\left(\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right|\right),$$

so the region $\mathcal{T} = (-\pi, -\theta_0) \cup (\theta_0, \pi)$ of Ω can be safely ignored.

3.1.4. Gaussian Approximation

Now we need to make a Gaussian approximation for the central region of Ω, i.e., for the integral over the region $(-\theta_0, \theta_0)$. We compute

$$\begin{aligned} b_{m,k}(n) &\sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0}\exp\Big(m\big(-L_{k,2}(\rho)\theta^2/2 - \mathbf{i}L_{k,3}(\rho)\theta^3/6 + \dots\big)\Big)\,d\theta \\ &\sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-m\,L_{k,2}(\rho)\theta^2/2}\,d\theta\Big(1 + O\Big(\frac{\rho^{1/k}}{k^2}\theta_0^3\Big)\Big). \end{aligned}$$

We recall $L_{k,3}(\rho) \sim \rho^{1/k}/k^2$, as proved in (12). As before, we have $\rho \sim (n/m)^k$, and we are working on the $(-\theta_0, \theta_0)$ portion of Ω. Putting these together, we get $\frac{\rho^{1/k}}{k^2}\theta_0^3 \sim \frac{n/m}{k^2}(n^{-4/9})^3 = O(n^{-1/3})$. Therefore, we get

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-mL_{k,2}(\rho)\theta^2/2}\,d\theta.$$

Again using (12), we have $L_{k,2}(\rho) \sim \rho^{1/k}/k$, and thus

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-m\rho^{1/k}\theta^2/(2k)}\,d\theta.$$

We do not change the first order asymptotics if we include the region $(-\infty, -\theta_0) \cup (\theta_0, \infty)$ in the contour, because we have

$$\int_{\theta_0}^{\infty} e^{-m\rho^{1/k}\theta^2/(2k)}\,d\theta = O(e^{-m\rho^{1/k}\theta_0^2/(2k)}) = O(e^{-n^{-1/9}/(2k)}),$$

so the integral over this region is exponentially small. So we extend the contour from $(-\theta_0, \theta_0)$ to $(-\infty, \infty)$, and we get

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\infty}^{\infty} e^{-m\rho^{1/k}\theta^2/(2k)}\,d\theta = \frac{(E_{k,0}(\rho))^m}{\rho^n}\frac{\sqrt{k}}{\sqrt{2\pi m\rho^{1/k}}}.$$

3.1.5. Conclusion of the Proof

Utilizing the form of $E_{k,0}(\rho)$, which was already given in (10), it follows that, for any (fixed) positive integers m and k, as $n \to \infty$, we have

$$b_{m,k}(n) \sim \left(\frac{em}{n}\right)^{kn} \sqrt{\frac{1}{2\pi n k^{m-1}} \left(\frac{m}{2\pi n}\right)^{(k-1)m}}. \tag{14}$$

This concludes the proof of Theorem 1.

4. Future Directions

Theorem 1 and Corollary 1 characterize the asymptotic properties of a general family of integer sequences, $a_{m,k}(n)$. We naturally view m and k as fixed, and we study the asymptotic analysis as $n \to \infty$. For a future direction of study, it would be natural to view the m and k as various kinds of functions of n, and to determine the asymptotic growth of $a_{k,m}(n)$ as m and k also grow (at various rates) with n. Such an investigation is beyond our scope in this relatively short treatment, but the recent work of Pemantle and Wilson [14] might be utilized for such a purpose.

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